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KP lumps in ferromagnets: a three-dimensional KdV–Burgers model

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Abstract

The three-dimensional long-wave approximation for electromagnetic waves in saturated ferromagnetic media is considered, taking into account damping and inhomogeneous exchange. The wave evolution is governed by a (3 + 1)-dimensional generalization of the Korteweg–de Vries, Burgers, Kadomtsev–Petviashvili and Zabolotskaya–Khokhlov equations. Neglecting the damping, we give plane-soliton and line-lump solutions, and show that they are unstable.

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1. Introduction

The long-wave limit of electromagnetic waves in a saturated ferromagnetic medium has been studied by several authors. One type of such waves is related to the so-called relativistic domain walls, which are strongly nonlinear waves propagating in ferromagnetic media. Explicit exact solutions of the Maxwell–Landau equations have been obtained [1]. The structure of these waves is close to that of the domain walls. Using a multiscale approach of 'long-wave' type, Nakata has shown that, in this limit, the evolution of this type of wave is governed by the modified Korteweg–de Vries (mKdV) equation. He obtained soliton-type solutions corresponding to one turn of the magnetization around the propagation direction [2]. Corrections to this type of model have been obtained, describing the effect of the damping [3], or taking into account the inhomogeneous exchange and the anisotropy [4], or the eventual presence of free charges, that leads to an additional damping [5], the antiferromagnetic character of the medium [6], or the effect of transverse perturbations on these structures [7].

The properties of the structures depend always on the angle φ between the propagation direction and the 'external' magnetic field. If this angle is zero, the equation describing the wave propagation is strongly modified. It becomes a nonlinear Schrödinger (NLS) equation in which the complex variable represents the component of the magnetization perpendicular to the propagation direction [8]. An asymptotic model rather close to this one, a derivative NLS-type equation, is obtained assuming that the inhomogeneous exchange interaction dominates [9].

Relativistic domain walls can thus be considered as the long-wave limit of electromagnetic waves with positive helicity. The long-wave limit of a wave with negative helicity has also been considered. It has been shown with the help of the multiscale expansion formalism that the propagation of this type of long wave is governed by the Korteweg–de Vries (KdV) equation [10]. Taking damping into account, a (2 + 1)-dimensional model of Burgers' type has been derived [11]. It admits explicit solutions describing the coalescence of n quasi-one-dimensional shock wave fronts. The higher order terms in the perturbative expansion that describes KdV soliton propagation have also been considered. The long-time propagation has been studied by means of a multi-time expansion. The propagation is governed by all equations of the KdV hierarchy [12]. Computation of the scale coefficients of the higher order time variables has shown that the KdV-type asymptotic is only valid when the angle between the propagation direction and the external magnetic field is large enough. Through the computation of the one-soliton solution of the whole KdV hierarchy, a maximum value of the soliton parameter has been determined, below which the KdV soliton conserves its properties during an infinite propagation time [13].

Wave interactions have been described. A wave of the KdV type can be partially reflected and partially transmitted through interaction with a relativistic domain wall [10]. A transverse instability of the relativistic domain walls is able to emit waves of the KdV type [7]. These waves can also be emitted by a short localized high-frequency wave packet. The emission is singular when the latter travels faster than the former and can be described by the Davey–Stewartson system [14, 15].

In section 2 of the present paper we derive a long-wave asymptotic, which describes the propagation of the KdV-type waves, taking into account damping, inhomogeneous exchange and two-dimensional transverse variations. It uses a (3 + 1)-dimensional multiscale expansion and a weak damping assumption. This model generalizes the KdV and Burgers' equations, and their two-dimensional versions, the Kadomtsev–Petviashvili (KP) and Zabolotskaya–Khokhlov equations. In section 3, neglecting the dispersion, a solution describing the coalescence of N shock fronts is written. Then neglecting the damping, plane-solitons and line-lumps solutions are given. We discuss the stability of these solutions with regard to slow transverse perturbations, and show that they are unstable. Perspectives and a conclusion are given in section 4.

2. The long-wave approximation

2.1. Multiple scales

The magnetic medium is described by the Landau–Lifschitz equation, which is, taking the damping and inhomogeneous exchange into account and neglecting anisotropy,

$$\partial_t \vec{M} = -\mu_0 \delta \vec{M} \wedge (\vec{H} + \beta \Delta \vec{M}) - \frac{\hat{\sigma}}{M_s} \vec{M} \wedge (\vec{M} \wedge \vec{H}). \tag{1}$$

δ is the gyromagnetic ratio, $\hat{\sigma}$ is a phenomenological positive damping constant, $M_s = \|\vec{M}\|$ is the saturated magnetization of the medium and β is the inhomogeneous exchange constant. The partial derivative of a function f relative to a variable X is denoted either by $\partial_X f$ or by f_X in the paper.

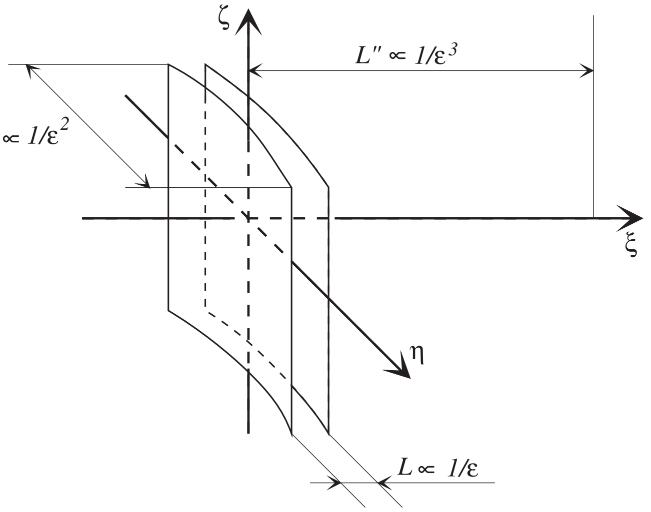

Figure 1. The various scales involved in the study of the transverse variations of the long waves.

The evolution of the magnetic field is described by the Maxwell equations, which reduce to

$$-\vec{\nabla}(\vec{\nabla} \cdot \vec{H}) + \Delta \vec{H} = \frac{1}{c^2} \partial_t^2 (\vec{H} + \vec{M}). \tag{2}$$

$\vec{\nabla}$ is the three-dimensional spatial gradient, and c is the light velocity based on the dielectric constant of the medium. We use a normalized form of equations (1) and (2), obtained using the definitions:

$$\vec{M}' = \frac{\delta\mu_0}{c}\vec{M} \qquad \vec{H}' = \frac{\delta\mu_0}{c}\vec{H} \qquad t' = ct \qquad \hat{\sigma}' = \frac{\hat{\sigma}c}{\delta^2\mu_0^2 M_s}. \tag{3}$$

This way, the constants $\mu_0\delta$, c and $\hat{\sigma}/M$ in equations (1) and (2) are replaced by 1, 1 and $\hat{\sigma}'$, respectively. The primes are omitted below.

We look for a weak amplitude and long-wave approximation using the following expansion:

$$\vec{M} = \vec{M}_0 + \varepsilon^2 \vec{M}_2 + \varepsilon^3 \vec{M}_3 + \cdots \tag{4}$$

where $\vec{M}_0 = \vec{m} = (m_x, m_t, 0) = m(\cos\varphi, \sin\varphi, 0)$ is a given constant vector and $\vec{M}_1, \vec{M}_2, \ldots,$ are functions of slow variables defined by

$$\xi = \varepsilon(x - Vt) \qquad \eta = \varepsilon^2(y - Ut) \qquad \zeta = \varepsilon^2(z - Wt) \qquad \tau = \varepsilon^3 t. \tag{5}$$

The scaling (5) is close to the one commonly used in the derivation of the KP equation in hydrodynamics [17]. The variable ξ of order ε describes the longitudinal shape of the pulse, with a typical length L of about $1/\varepsilon$ in normalized units. The transverse space variables η and ζ account for the transverse variations of the wave. These variables are slower than ξ, so that the direction of the plane of the wave is fixed. The transverse extension L' of the wave is about $1/\varepsilon^2$ in normalized units, very large with regard to its length, as illustrated in figure 1. The time variable τ describes the long-time or long-distance propagation: the propagation distance L'' is about $1/\varepsilon^3$. A remarkable feature of the scaling (5) is the introduction of transverse velocity components U and W. They describe the fact that the wave velocity is not exactly perpendicular to the wave plane. All profiles $\vec{M}_n, \vec{H}_n$ $(n \geqslant 2)$ are assumed to vanish as $\xi \longrightarrow +\infty$. $\vec{H}$ is expanded in the same way as $\vec{M}$. At order zero it is found that $\vec{H}_0$ must

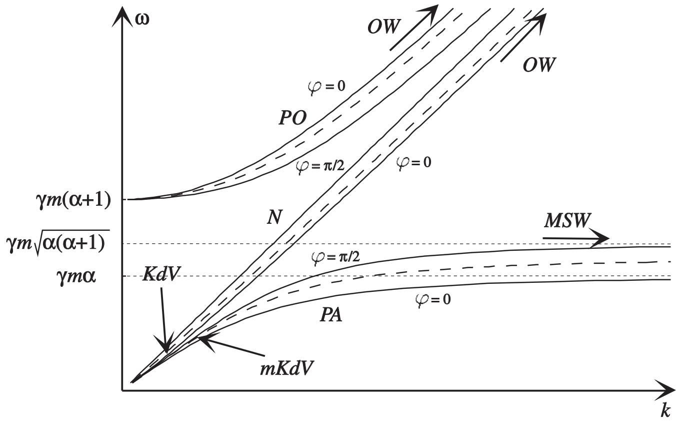

Figure 2. Dispersion relation of the electromagnetic waves in ferromagnetic media, with the indication of the various modes, and for several values of the angle φ between the propagation direction and the applied magnetic field. PO: optical with positive helicity, PA: acoustic with positive helicity, N: with negative helicity, KdV, mKdV: long-wave modes governed by the KdV and by the modified KdV equations, respectively, MSW: magnetostatic waves, OW: optical waves.

be collinear to $\vec{M}_0$. We write $\vec{H}_0 = \alpha\vec{m}$ where α is some constant. Stability of the uniform background requires that α is positive. We add a weak damping approximation, writing the damping constant $\hat{\sigma}$ as

$$\hat{\sigma} = \varepsilon\sigma. \tag{6}$$

2.2. Polarization and velocity

The first non-trivial equation of the perturbative scheme is obtained at order ε^4 in the Maxwell equation (2) and ε^2 in the Landau equation (1). It yields polarization vectors

$$\vec{H}_2 = \vec{h}_1 g \qquad \vec{M}_2 = \vec{m}_1 g \tag{7}$$

where

$$\vec{h}_1 = \begin{pmatrix} \mu m_x \\ (1+\alpha)m_t \\ 0 \end{pmatrix} \qquad \text{and} \qquad \vec{m}_1 = \begin{pmatrix} -\mu m_x \\ -\gamma(1+\alpha)m_t \\ 0 \end{pmatrix}. \tag{8}$$

We use the shortcuts $\gamma = 1 - \frac{1}{V^2}$, $\mu = 1 + \alpha\gamma$.

At the following order, we get a compatibility condition yielded by the dot product of the Landau equation (1) by $\vec{m}$. It gives the value of the velocity

$$V = \sqrt{\frac{\alpha + \sin^2\varphi}{\alpha + 1}}. \tag{9}$$

Thus the considered wave belongs to the propagation mode characterized by this velocity, which is known to support KdV solitons for certain scales [10]. This long-wave propagation mode can be considered as the small wave vector limit of an oscillating wave, as can be seen from the dispersion relation drawn in figure 2. Both long-wave propagation modes able to propagate in the medium appear in the plot of the dispersion relation. Expression (9) of the velocity V allows us to identify the mode belonging to the branch with negative helicity N, labelled KdV. The other mode labelled mKdV corresponds to relativistic domain walls (see the introduction).

Expression of the third-order terms of the magnetic field $\vec{H}_3$ and magnetization $\vec{M}_3$ are also found. They are

$$\vec{M}_3 = \vec{h}_1 f + \begin{pmatrix} \frac{-(1+\alpha)m_t}{V^2}\int_{-\infty}^{\xi} g_\eta \\ \frac{(\alpha-1)m_x}{V^2}\int_{-\infty}^{\xi} g_\eta - \frac{2(1+\alpha)m_t}{\mu V^3}\int_{-\infty}^{\xi}(U\partial_\eta + W\partial_\zeta)g \\ \frac{\gamma V m_x}{m_t} g_\xi - \frac{m_x}{V^2}\int_{-\infty}^{\xi} g_\zeta \end{pmatrix} \tag{10}$$

and

$$\vec{H}_3 = \vec{h}_1 f + \begin{pmatrix} 0 \\ \frac{-\alpha m_x(\gamma+1)}{\gamma V^2}\int_{-\infty}^{\xi} g_\eta - \frac{2\alpha(1+\alpha)m_t}{\mu V^3}\int_{-\infty}^{\xi}(U g_\eta + W g_\zeta) \\ \frac{-V m_x}{m_t} g_\xi - \frac{\alpha m_x}{V^2}\int_{-\infty}^{\xi} g_\zeta \end{pmatrix} \tag{11}$$

where the third-order amplitude f is a function of (ξ, η, ζ, τ) to be determined.

At the following order, ε^6 for equation (2) and ε^4 for equation (1), the compatibility condition $\vec{m} \cdot \vec{M}_3 = 0$ can be written in the form

$$(a + bU)g_\eta + cWg_\zeta = 0 \tag{12}$$

where the constants a, b, c have explicit expressions involving α, m_x and m_t. If the quantities $a + bU$ and cW are not both zero, the solution of equation (12) is an arbitrary function of $(cW\eta - (a+bU)\zeta)$. Assuming that $\vec{M}_3$ vanishes in any direction in the (η, ζ) plane it is found that $a + bU$ and W must be zero. The direction of the transverse velocity is fixed by $W = 0$. The velocity belongs to the plane containing the normal x to the plane of the wave, and the static magnetization $\vec{m}$. The transverse velocity U is then

$$U = \frac{\gamma m_t}{m_x} V = \gamma V \tan\varphi. \tag{13}$$

Especially, it is zero when $m_t = 0$, i.e. when the plane of the wave is perpendicular to the static magnetization, the symmetry of rotation around this direction must be conserved, and this implies that the velocity is exactly parallel to the static magnetization. In [11] the transverse velocity component was omitted, and it was found that the (2 + 1)-dimensional asymptotic model of Burgers type could be derived only when the transverse modulation is perpendicular to the magnetization, which is equivalent to considering the variable ζ only, and not η.

The transverse velocity originates in the anisotropy of the medium induced by the external field. It can be recovered as a long-wave limit of the transverse component of the group velocity of an oscillating wave, as follows. The dispersion relation of electromagnetic waves in ferromagnetic media is

$$\mu^2 m^2 \cos^2\varphi + \gamma\mu(1+\alpha)m^2 \sin^2\varphi = \gamma^2\omega^2. \tag{14}$$

To compute the transverse group velocity, the direction of the wave vector $\vec{k}$ must vary freely, thus the angle φ between $\vec{k}$ and the static magnetization $\vec{m}$ also must be free. Therefore, we set $\varphi = \varphi_0 - \psi$ where φ_0 is fixed, and ψ is small. The transverse group velocity is then

$$V_t = \frac{1}{k}\partial_\psi \omega(k, \psi)\bigg|_{\psi=0}. \tag{15}$$

Explicit computation shows that the long-wave limit of V_t, on the N branch of the dispersion relation, i.e. with $\omega = kV$, exactly coincides with the transverse velocity U given by equation (13), which confirms the above interpretation. According to expression (13), the transverse velocity U changes its sign with both transverse m_t and longitudinal m_x components of the magnetization. The orientation of the transverse plane is fixed by the direction of the static magnetization $\vec{m}$. The direction of U does not change with regard to $\vec{m}$ when m_t changes its sign. Changing the sign of m_x is equivalent to changing the propagation direction into the opposite. The direction of the transverse velocity is changed into the opposite when the wave propagates in the reverse direction, so that the ray is not modified.

2.3. Derivation of a nonlinear evolution equation

Taking into account expressions (12) and (13) of the velocity components V and U, expressions (10) and (11) of the third-order field and magnetization become

$$\vec{M}_3 = \vec{h}_1 f + \begin{pmatrix} \frac{-(1+\alpha)m_t}{V^2}\int_{-\infty}^{\xi} g_\eta \\ \frac{(1+\alpha)m_x}{V^2}\int_{-\infty}^{\xi} g_\eta \\ \frac{\gamma V m_x}{m_t} g_\xi - \frac{m_x}{V^2}\int_{-\infty}^{\xi} g_\zeta \end{pmatrix} \tag{16}$$

$$\vec{H}_3 = \vec{h}_1 f + \begin{pmatrix} 0 \\ \frac{-\alpha m_x}{\gamma V^4}\int_{-\infty}^{\xi} g_\eta \\ \frac{-V m_x}{m_t} g_\xi - \frac{\alpha m_x}{V^2}\int_{-\infty}^{\xi} g_\zeta \end{pmatrix}. \tag{17}$$

From the Maxwell equation (2) at order ε^6 we deduce

$$M_4^x = -H_4^x - \frac{(1+\alpha)m_t}{V^2}\int_{-\infty}^{\xi} f_\eta + \frac{m_x}{V m_t} g_\zeta + \frac{m_x}{\gamma V^2}(\alpha(1-\gamma)^2 - \gamma\mu)\int_{-\infty}^{\xi}\int_{-\infty}^{\xi} g_{\eta\eta} + \frac{(1+\alpha)m_x}{V^2}\int_{-\infty}^{\xi}\int_{-\infty}^{\xi} g_{\zeta\zeta} \tag{18}$$

$$M_4^y = -\gamma H_4^y + \frac{\mu m_x}{V^2}\int_{-\infty}^{\xi} f_\eta + \frac{m_t}{V^2}(2\alpha(1-\gamma) + \gamma(\mu - 2(1+\alpha)))\int_{-\infty}^{\xi}\int_{-\infty}^{\xi} g_{\eta\eta} + \frac{(1+\alpha)m_t}{V^2}\int_{-\infty}^{\xi}\int_{-\infty}^{\xi} g_{\zeta\zeta} + \frac{2(1-\gamma)(1+\alpha)m_t}{V}\int_{-\infty}^{\xi} g_\tau. \tag{19}$$

We define a fourth-order amplitude ψ by

$$H_4^x = \mu m_x \psi \tag{20}$$

then the z-component of the Landau equation (1) at order ε^4 allows us to compute H_4^y as

$$H_4^y = m_t(1+\alpha)\psi - \frac{\alpha m_x}{\gamma V^4}\int_{-\infty}^{\xi} f_\eta + \frac{2\alpha(1+\alpha)m_t}{\mu V^3}\int_{-\infty}^{\xi} g_\tau - \frac{(1+\alpha)}{\mu V} g_\zeta - \frac{\sigma V m_t}{\mu V^2} g_\xi + \frac{(1+\alpha)m_t}{V^2} g^2 + \left(\frac{\gamma V^2}{\mu m_t} - \frac{\beta m_t}{\mu V^2}\right) g_{\xi\xi} - \frac{\alpha m_t}{\gamma\mu V^4}(\alpha(1-3\gamma+\gamma^2) - \gamma)\int_{-\infty}^{\xi}\int_{-\infty}^{\xi} g_{\eta\eta}. \tag{21}$$

The nonlinear evolution equation is given by the projection along the external magnetization, i.e. the dot product by $\vec{m}$, of the Landau equation (1) at order ε^5. Taking into account the solvability conditions at previous order,

$$\vec{m}\cdot\vec{M}_3 = \vec{m}\cdot\vec{M}_2 = \vec{0} \tag{22}$$

and noting that many terms in the expansion of the outer product are orthogonal to $\vec{m}$, the solvability condition reduces to

$$V\partial_\xi \vec{m}\cdot\vec{M}_4 = \vec{m}\cdot(\vec{M}_2 \wedge \vec{H}_3 + \vec{M}_3 \wedge \vec{H}_2). \tag{23}$$

Equation (23) is made explicit using expressions (16)–(21) of the field and magnetization components. The terms involving the third-order amplitude ψ vanish using the expression of the velocity V. The terms involving the second-order amplitude f vanish using the expression of the transverse velocity U, W. Equation (23) reduces then to

$$Ag_\tau + Bg_{\xi\xi\xi} + Cg_{\xi\xi} + Dgg_\xi + E\int_{-\infty}^{\xi} g_{\eta\eta} + F\int_{-\infty}^{\xi} g_{\zeta\zeta} = 0 \tag{24}$$

in which the real coefficients A, B, C, D, E and F are given by

$$A = \frac{2(1+\alpha)m_t^2}{\mu V^2} \qquad B = \frac{-\gamma V}{\mu}\left(\gamma V^2 - \frac{\beta m_t^2}{V^2}\right) \qquad C = \frac{\sigma\gamma m_t^2}{\mu} \tag{25}$$

$$D = \frac{3\mu m_x^2}{V} \qquad E = \frac{\alpha m_t^2}{\mu V^5} \qquad F = \frac{(1+\alpha)m^2}{V}. \tag{26}$$

Equation (24) is a (3 + 1)-dimensional form of the KP equation, including a damping term that recalls the Burgers equation.

3. A (3 + 1)-dimensional KP–Burgers model

3.1. Coalescence of shock wave fronts

The model equation (24) is related to the KdV and Burgers equations, and to their (2 + 1)-dimensional generalizations, the KP equation and the Zabolotskaya–Khokhlov one, respectively. We give now some detail on this point. The change of variables

$$X = \xi + a(-\eta\sin\theta + \zeta\cos\theta) - \frac{a^2}{A}(E\sin^2\theta + F\cos^2\theta)\tau \tag{27}$$

$$Y = \eta\cos\theta + \zeta\sin\theta - 2a\sin\theta\cos\theta\frac{F-E}{A}\tau \tag{28}$$

reduces the (3 + 1)-dimensional KP–Burgers' equation (24) to

$$(Ag_\tau + Bg_{XXX} + Cg_{XX} + Dgg_X)_X = -(E\cos^2\theta + F\sin^2\theta)g_{YY}. \tag{29}$$

Dropping the transverse variable Y, equation (29) reduces to

$$Ag_\tau + Bg_{XXX} + Cg_{XX} + Dgg_X = 0. \tag{30}$$

Neglecting the dispersion coefficient B, the (1 + 1)-dimensional reduction (30) of equation (24) is the Burgers equation. It has been derived in [11] using a different scaling: the slow variables considered were

$$\xi = \varepsilon(x - Vt) \qquad \tau = \varepsilon^2 t, \tag{31}$$

no weak damping assumption was made, and the expansion of the field began at order ε instead of ε^2 in equation (4). Assuming a higher amplitude and strong damping, the nonlinear and diffusive terms appear sooner, so that they involve smaller propagation time, about $1/\varepsilon^2$ instead of $1/\varepsilon^3$. The dispersion term arises only for the latter order of magnitude of the propagation time and thus is negligible.

Recall that the Burgers equation can be linearized by means of an adequate transformation, called the Hopf–Cole one [20]. The change of variables

$$g = \frac{-C}{D}u \qquad \tau = \frac{-A}{C}T \tag{32}$$

reduces equation (30) to

$$u_T - u_{XX} + uu_X = 0. \tag{33}$$

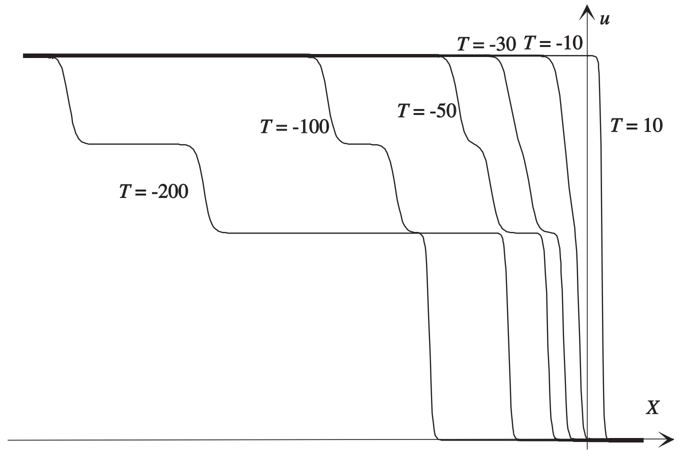

Figure 3. Coalescence of shock waves described by the Burgers' equation, in normalized units. $(f_i) = (1, 1, 1)$, $(v_i) = (0.7, 1, 1.3)$.

The Hopf–Cole transformation can then be written as

$$u = -2\partial_X \ln U \tag{34}$$

and F is a solution of the heat equation:

$$U_T = U_{XX}. \tag{35}$$

If we choose for U a linear combination of exponentials, we obtain the following particular solution of equation (29) with $B = 0$:

$$g = \frac{-2C}{D} \frac{\sum_{i=1}^{n} f_i v_i \,\mathrm{e}^{\varphi_i}}{1 + \sum_{i=1}^{n} f_i \,\mathrm{e}^{\varphi_i}} \tag{36}$$

where the phases φ_i are

$$\varphi_i = -v_i \left[\xi + a(-\eta \sin\theta + \zeta \cos\theta) - \frac{1}{A}[a^2(E \sin^2\theta + F \cos^2\theta) - v_i C]\tau \right] \tag{37}$$

and v_i, f_i are arbitrary real parameters. Expression (37) describes the propagation of n shock profiles, and their coalescence, cf figure 3.

Still neglecting the dispersion coefficient B, the (2 + 1)-reduction (29) of equation (24) is the (2 + 1)-dimensional Burgers equation, also known as the Zabolotskaya–Khokhlov equation. It coincides exactly with the equation of this type found in [11] when the transverse variable Y is ζ, i.e. when $\theta = \pi/2$. The introduction of the transverse velocity U allowed us to generalize the result of [11]. The (2 + 1)-dimensional Burgers equation is not integrable. It can be written using a generalization of Hirota's bilinear formalism. This allows us to derive explicit solutions, which describe the coalescence of n interacting quasi-one-dimensional wave fronts, but only if they propagate in the same direction [21]. The solutions obtained this way are essentially the same as that given by equation (36). The bilinear form given in [21] easily generalizes to (3 + 1) dimensions, but according to the rather deceiving (2 + 1)-dimensional result, it is doubtful that new solutions of equation (24) with $B = 0$ can be obtained this way.

3.2. Plane-solitons

If we neglect the damping, i.e. if we set $\sigma = 0$, then C is zero and equation (24) reduces to the (3 + 1)-dimensional KP equation

$$Ag_\tau + Bg_{\xi\xi\xi\xi} + Dgg_\xi + E\int_{-\infty}^{\xi} g_{\eta\eta} + F\int_{-\infty}^{\xi} g_{\zeta\zeta} = 0. \tag{38}$$

Then the (1 + 1)-dimensional reduction (30) of equation (24) or (38) is the KdV equation. It coincides exactly with the KdV equation derived in [10] if the propagation direction is the x-axis, i.e. $a = 0$, and if the exchange constant β involved in the dispersion coefficient B is neglected. The one-soliton solution of the KdV equation yields the following solution of equation (38):

$$g = \frac{12k^2B}{D}\mathrm{sech}^2 k\left(\xi + a(\zeta\cos\theta - \eta\sin\theta) - [a^2(E\sin^2\theta + F\cos^2\theta) + 4k^2B]\frac{\tau}{A}\right). \tag{39}$$

a, θ and k are arbitrary real parameters. We call these quasi-one-dimensional solutions plane-solitons by analogy with the line-solitons of the KP equation. k characterizes the underlying KdV soliton and a, θ determine the direction of the plane.

Indeed, if we still neglect the damping ($C = 0$), the (2 + 1)-dimensional reduction (29) of equation (24) is the KP equation, i.e. the typical (2 + 1)-dimensional generalization of the KdV equation. It was first derived in order to describe the transverse stability of hydrodynamical surface waves [16], and it is integrable by means of the IST method [18]. Soliton solutions in the sense of the IST transforms are the line-solitons, whose (3 + 1)-dimensional analogue is the plane-soliton (39). Studies of the stability of the line-solitons of the KP equation are given in [16] and [18]. They show that the line-soliton solution is stable with regard to slow transverse perturbations when the product $B\mathcal{E}$ of the coefficient B of the dispersion term and the coefficient $\mathcal{E}$ of the transverse variations term is positive ($\mathcal{E} = E\sin^2\theta + F\cos^2\theta$ in the case of the equation (29)), and unstable when $B\mathcal{E}$ is negative. The KP equation is called KP II in the former case (stable line solitons) and KP I in the latter (unstable line solitons). Since

$$\gamma = \frac{-\cos^2\varphi}{\alpha + \sin^2\varphi} \tag{40}$$

is negative and

$$\mu = \frac{(\alpha + 1)\sin^2\varphi}{\alpha + \sin^2\varphi} \tag{41}$$

is positive, it is seen that the coefficients E and F of the terms that describe the transverse variations in equation (38) or (24) are both positive. The dispersion coefficient B is negative for any value of the positive exchange constant β. Thus the KP equation, equation (29) with $C = 0$, is of type I for any value of the angle θ. The corresponding line-soliton solutions are unstable with regard to slow transverse perturbations [18]. The proof straightforwardly generalizes to the (3 + 1)-dimensional case of the above plane-soliton (39), which is thus unstable with regard to slow transverse perturbations.

Due to the complete integrability of the KP equation, it is possible to find an N-line soliton solution. It can be obtained either by means of the Hirota bilinear method or by means of the IST method. From the N-line soliton solution can be straightforwardly deduced an N-plane soliton solution of the three-dimensional KP equation (38), but with the constraint that the propagation directions of all plane-solitons are coplanar. The Hirota bilinear form of the KP equation, given in [19], generalizes easily to (3 + 1) dimensions. It is

$$\left(AD_\xi D_\tau + BD_\xi^4 + ED_\eta^2 + FD_\zeta^2\right) f\cdot f = 0 \tag{42}$$

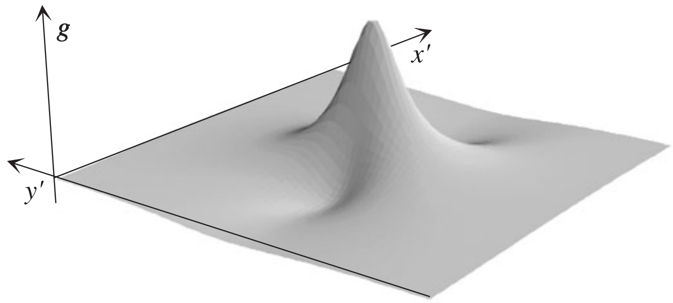

Figure 4. Lump solution of the KP equation ($p_r = 0.2$, $p_i = 1.5$).

where the bilinear operators D_ξ^n are defined by

$$D_\xi^n f_1 \cdot f_2 = (\partial_\xi - \partial_{\xi'})^n f_1(\xi) f_2(\xi')|_{\xi'=\xi} \tag{43}$$

and the amplitude g is recovered from f through

$$g = \left(\frac{12B}{D} \ln f\right)_{\xi\xi}. \tag{44}$$

It should *a priori* be possible to derive an N-plane soliton solution without the above-mentioned constraint using this formalism. This is left for further study.

3.3. Line-lump

The KP I equation also admits localized algebraically decaying solutions called lumps, as shown in figure 4. Using the expression of the lump solution of KP I given in [19], we obtain the following expression of a line-lump solution of the three-dimensional KP equation (38):

$$g = \frac{24B}{D} \frac{\left[-(x' + p_r y')^2 + p_i^2 y'^2 + 3/p_i^2\right]}{\left[(x' + p_r y')^2 + p_i^2 y'^2 + 3/p_i^2\right]^2} \tag{45}$$

with

$$x' = X - \left(p_r^2 + p_i^2\right) \frac{B}{A} \tau \tag{46}$$

and

$$y' = \sqrt{\frac{-B}{E\cos^2\theta + F\sin^2\theta}} Y + 2p_r \frac{B}{A} \tau. \tag{47}$$

Using equations (27) and (28), the variables x' and y' can be written more explicitly as

$$x' = \xi + a(-\eta \sin\theta + \zeta \cos\theta) - \left[a^2(E\sin^2\theta + F\cos^2\theta) + \left(p_r^2 + p_i^2\right) B\right] \frac{\tau}{A} \tag{48}$$

and

$$y' = \frac{\sqrt{-B}}{\sqrt{E\cos^2\theta + F\sin^2\theta}} (\eta\cos\theta + \zeta\sin\theta) - \left[\frac{2a\sin\theta\cos\theta\sqrt{-B}(F - E)}{\sqrt{E\cos^2\theta + F\sin^2\theta}} - 2p_r B\right] \frac{\tau}{A}. \tag{49}$$

An N lumps solution of the KP I equation is given in [19]. An N line-lumps solution can be deduced from it, in the special case where the axes of the N interacting line-lumps are parallel. A general N line-lumps solution might eventually be derived using the Hirota bilinear form (42), (44) and the procedure of [19]. It is left for further study.

3.4. Stability of the line-lump

It has been shown in [22] that the line-lump solution (45) of the three-dimensional KP I equation is unstable with regard to transverse perturbation, using a linear stability analysis at large wavelengths. We give here another proof of this instability, which considers a slow transverse perturbation of the soliton parameter, following the approach of [18] for the line-solitons of KP. Firstly the (3 + 1)-dimensional KP equation (38) is written in normalized form as

$$(u_T + 6uu_X + u_{XXX})_X = u_{YY} + u_{ZZ}. \tag{50}$$

(Avoid confusions between the notation of this subsection and the previous ones.) With this normalization, the line-lump is

$$u^{(0)} = \frac{4\left[-(x' + p_r y')^2 + p_i^2 y'^2 + 3\big/p_i^2\right]}{\left[(x' + p_r y')^2 + p_i^2 y'^2 + 3\big/p_i^2\right]^2} \tag{51}$$

with

$$x' = X - \left(p_r^2 + p_i^2\right) T - X^{(0)}(\delta T, \delta Z) \tag{52}$$

$$y' = Y + 2p_r T - Y^{(0)}(\delta T, \delta Z). \tag{53}$$

We assume without loss of generality that p_i is positive, and $|p|^2 = p_r^2 + p_i^2$. δ is a small parameter, thus $X^{(0)}$ and $Y^{(0)}$ vary slowly with T and Z and describe a slow transverse perturbation. We denote by τ and ζ the slow variables δT and δZ, respectively. The field u is expanded in a power series of δ as

$$u = u^{(0)} + \delta u^{(1)}(x', y', \tau, \zeta) + \delta^2 u^{(2)}(x', y', \tau, \zeta) + \cdots. \tag{54}$$

The expansion (54) is substituted into the (3 + 1)-dimensional KP equation (50). For each $n \geqslant 1$, at order δ^n it yields an equation for $u^{(n)}$ of the form

$$\mathcal{L}u^{(n)} = \mathcal{F}^{(n)} \tag{55}$$

where the operator $\mathcal{L}$ is defined by

$$\mathcal{L}v = -|p|^2 v_X + 2p_r v_Y + 6(u^{(0)}v)_X + v_{XXX} - \int_{-\infty}^{X} v_{YY}. \tag{56}$$

At order δ, the right-hand side of equation (55) is

$$\mathcal{F}^{(1)} = u_X^{(0)} X_\tau^{(0)} + u_Y^{(0)} Y_\tau^{(0)}. \tag{57}$$

Using the identities

$$\mathcal{L}\left(2u^{(0)} + Xu_X^{(0)}\right) = 2|p|^2 u_X^{(0)} - 6p_r u_Y^{(0)} + 4\int_{-\infty}^{X} u_{YY}^{(0)} \tag{58}$$

$$\mathcal{L}\left(Yu_Y^{(0)}\right) = 2p_r u_Y^{(0)} - 2\int_{-\infty}^{X} u_{YY}^{(0)} \tag{59}$$

and

$$\mathcal{L}\left(Yu_X^{(0)}\right) = 2p_r u_X^{(0)} - 2u_Y^{(0)} \tag{60}$$

we find the explicit solution

$$u^{(1)} = \frac{1}{2p_i^2}\left[\left(X_\tau^{(0)} + p_r Y_\tau^{(0)}\right)\left(2u^{(0)} + Xu_X^{(0)} + 2Yu_Y^{(0)}\right) - \left(p_r X_\tau^{(0)} + |p|^2 Y_\tau^{(0)}\right) Y u_X^{(0)}\right] \tag{61}$$

of equation (55) for $n = 1$.

At order δ^2 we find

$$\mathcal{F}^{(2)} = -6u^{(1)}u_X^{(1)} + \int_{-\infty}^{X} u_{\zeta\zeta}^{(0)} - u_\tau^{(1)} \tag{62}$$

in which

$$u_{\zeta\zeta}^{(0)} = u_{XX}^{(0)}\left(X_\zeta^{(0)}\right)^2 + 2u_{XY}^{(0)}X_\zeta^{(0)}Y_\zeta^{(0)} + u_{YY}^{(0)}\left(Y_\zeta^{(0)}\right)^2 - u_X^{(0)}X_{\zeta\zeta}^{(0)} - u_Y^{(0)}Y_{\zeta\zeta}^{(0)}. \tag{63}$$

The adjoint operator $\mathcal{L}^*$ of $\mathcal{L}$ is defined by

$$\mathcal{L}^*v = |p|^2 v_X - 2p_r v_Y - 6u^{(0)}v_X - v_{XXX} + \int_{-\infty}^{X} v_{YY}. \tag{64}$$

Therefore $\mathcal{L}^*u^{(0)} = 0$ and, computing the scalar product $(u^{(0)}|\mathcal{L}u^{(2)})$, we find that the condition

$$\iint_{\mathbb{R}^2} \mathrm{d}x\,\mathrm{d}y\, u^{(0)}\mathcal{F}^{(2)} = 0 \tag{65}$$

must be satisfied. $\mathcal{F}^{(2)}$ is expanded into a sum

$$\begin{aligned}\mathcal{F}^{(2)} = a_1X_{\zeta\zeta}^{(0)} + a_2X_{\tau\tau}^{(0)} + a_3Y_{\zeta\zeta}^{(0)} + a_4Y_{\tau\tau}^{(0)} + a_5X_\tau^{(0)}Y_\tau^{(0)} + a_6\left(X_\tau^{(0)}\right)^2 + a_7\left(Y_\tau^{(0)}\right)^2 \\ + a_8X_\zeta^{(0)}Y_\zeta^{(0)} + a_9\left(X_\zeta^{(0)}\right)^2 + a_{10}\left(Y_\zeta^{(0)}\right)^2.\end{aligned} \tag{66}$$

The integrals $\iint_{\mathbb{R}^2} \mathrm{d}x\,\mathrm{d}y\; a_j$ are computed using a formal computation software. After division by a factor $-2\pi/(3p_i)$ condition (65) reduces to

$$4p_i^2X_{\zeta\zeta}^{(0)} + X_{\tau\tau}^{(0)} + 4p_i^2p_rY_{\zeta\zeta}^{(0)} + p_rY_{\tau\tau}^{(0)} = 0. \tag{67}$$

This proves that the line-lump (45) is unstable with regard to slowly varying transverse perturbations.

Numerical investigation has shown the relation between the instability of the line-soliton of KP and the existence of the lump solution. Indeed, the evolution of a transversely perturbed line-soliton leads to the formation of lumps [23]. The same phenomenon occurs in (3 + 1) dimensions: a distorted line-lump evolves into localized three-dimensional structures [24]. It has been shown that such a structure is not stable but collapses. However, it collapses very slowly with regard to the rate at which it forms from the instability of the line-lump. Hence, although unstable, these three-dimensional soliton-like structures are apparently stable in the simulations [24]. They can be stabilized by a fifth-order dispersion term [25]. A refinement of the model presented here could take such a higher order dispersion term into account. On the other hand, the dissipation slows down the collapse process [25]. It can thus be expected that an apparently self-similar, soliton-type propagation can arise from an adequate balance between dissipation and self-focusing, as has been shown in the case of envelope solitons in yttrium iron garnet thin film [26]. This is left for further study.

4. Conclusion

A (3 + 1)-dimensional KP–Burgers model has been derived. It describes the effect of damping and of transverse variations on the KdV-type solitons able to propagate in a saturated ferromagnetic medium. This model generalizes the KdV, Burgers, KP and Zabolotskaya–Khokhlov equations, some of which had already been found as rougher approximations of the considered problem.

Although a more complete analysis of this system is left for further study, we give explicitly some particular solutions, in the two special situations where either damping or dispersion is negligible. When dispersion is neglected, the coalescence of N shock fronts is described.

Neglecting the damping, the derived model reduces to a (3 + 1)-dimensional KP equation of type I. It admits quasi-one-dimensional solutions, that we have called plane-solitons, and quasi-two-dimensional solutions, that we called line-lumps. All these solutions are unstable with regard to slow transverse perturbations. Although the (3 + 1)-dimensional KP equation does not possess any fundamentally stable spatially localized solution, three-dimensional localized structures are known, which can appear as stable during a limited propagation time. Refinement of the model and of its study, in order to enhance the stability of these structures, can be envisaged in the future.

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